Chapter 8

DISCRETE PROBABILITY DISTRIBUTIONS

8.2. (b) The binomial probability distribution with \( n = 3 \) and \( p = 0.4 \) is

\[
f(x) = \binom{3}{x} (0.4)^x (0.6)^{3-x}, \quad \text{for } x = 0, 1, 2, 3.
\]

Now \( P\left(X = \frac{3}{2}\right) = f\left(\frac{3}{2}\right) = 0; \) because a random variable \( X \) with a binomial distribution takes only one of the integer values 0, 1, 2, ..., \( n \).

\[
P(X=2) = \binom{3}{2} (0.4)^2 (0.6)^{3-2} = 0.288;
\]

\[
P(X\leq2) = \sum_{x=0}^{2} \binom{3}{x} (0.4)^x (0.6)^{3-x}
\]

\[
= \binom{3}{0}(0.4)^0(0.6)^3 + \binom{3}{1}(0.4)^1(0.6)^2 + \binom{3}{2}(0.4)^2(0.6)^1
\]

\[
= 0.216 + 0.432 + 0.288 = 0.936;
\]

\[
P(X=-2) = f(-2) = 0; \) because a random variable \( X \) with a binomial distribution takes only one of the non-negative integer values 0, 1, 2, 3, ..., \( n \).

\[
P(X\geq2) = \sum_{x=2}^{3} \binom{3}{x} (0.4)^x (0.6)^{3-x}
\]

\[
= \binom{3}{2} (0.4)^2 (0.6)^1 + \binom{3}{3} (0.4)^3 (0.6)^0
\]

\[
= 0.288 + 0.064 = 0.352.
\]
8.3. (a) We observe that

(i) there are two possible outcomes, i.e. we will get or will not get a "5 or 6";

(ii) the probability of getting a "5 or 6" in each trial is \( p = \frac{2}{6} \).

(iii) the successive trials are independent, and

(iv) there are 5 trials.

Therefore the binomial distribution with \( n = 5 \) and \( p = \frac{1}{3} \) is appropriate.

Let \( X \) denote the number of successes. Then

(i) \[ P(\text{no success}) = P(X=0) = \binom{5}{0} \left( \frac{1}{3} \right)^0 \left( \frac{2}{3} \right)^5 = \frac{32}{243} = 0.1317; \]

(ii) \[ P(\text{at least 2 successes}) = P(X\geq2) = 1 - P(X<2) = 1 - \sum_{x=0}^{1} \binom{5}{x} \left( \frac{1}{3} \right)^x \left( \frac{2}{3} \right)^{5-x} \]

\[ = 1 - \left[ \binom{5}{0} \left( \frac{1}{3} \right)^0 \left( \frac{2}{3} \right)^5 + \binom{5}{1} \left( \frac{1}{3} \right)^1 \left( \frac{2}{3} \right)^{5-1} \right] \]

\[ = 1 - \left[ \frac{32}{243} + \frac{80}{243} \right] = \frac{131}{243} = 0.5391 \]

(iii) \[ P(\text{at least 1 but not more than 3}) = P(1\leq X\leq3) = \sum_{x=1}^{3} \binom{5}{x} \left( \frac{1}{3} \right)^x \left( \frac{2}{3} \right)^{5-x} \]

\[ = \binom{5}{1} \left( \frac{1}{3} \right)^1 \left( \frac{2}{3} \right)^4 + \binom{5}{2} \left( \frac{1}{3} \right)^2 \left( \frac{2}{3} \right)^3 + \binom{5}{3} \left( \frac{1}{3} \right)^3 \left( \frac{2}{3} \right)^2 \]

\[ = \frac{80}{243} + \frac{80}{243} + \frac{40}{243} = \frac{200}{243} = 0.8230. \]
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(b) (i) Let X denote the number of successes. Then using the binomial distribution, we get

\[ P(X=3) = \binom{8}{3} (0.4)^3 (0.6)^{8-3} \]
\[ = (56) (0.064) (0.07776) = 0.2787 \]

(ii) In the binomial distribution with \( n = 6 \), 2 failures imply four successes, i.e. \( P(2 \text{ failures in } 6 \text{ trials}) = P(4 \text{ successes in } 6 \text{ trials}) \). Thus

\[ P(X=4) = \binom{6}{4} (0.6)^4 (0.4)^2 \]
\[ = (15) (0.1296) (0.16) = 0.3110 \]

(iii) Let X denote the number of successes. Then

\[ P(X \leq 2) = \sum_{x=0}^{2} \binom{9}{x} (0.4)^x (0.6)^{9-x} \]
\[ = \binom{9}{0} (0.4)^0 (0.6)^9 + \binom{9}{1} (0.4)(0.6)^8 + \binom{9}{2}(0.4)^2(0.6)^7 \]
\[ = 0.010078 + 0.060466 + 0.161243 = 0.2318 \]

8.4. (a) Let X denote the number of heads when 6 coins are tossed. Then

(i) \( P(X=\text{exactly } 4 \text{ heads}) = \binom{6}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = 15 \cdot \frac{1}{64} = \frac{15}{64} \)

(ii) \( P(X=\text{not more than } 4 \text{ heads}) = 1 - P(X>4) \)
\[ = 1 - \sum_{x=5}^{6} \binom{6}{x} \left(\frac{1}{2}\right)^6 \]
\[ = 1 - \left(\frac{6}{64} + \frac{1}{64}\right) = \frac{57}{64} \]

(b) Let X denote the number of heads in a single toss of 6 fair coins. Then

(i) \( P(X \geq 2) = 1 - P(X < 2) \)
\[ = 1 - \sum_{x=0}^{1} \binom{6}{x} \left(\frac{1}{2}\right)^6 = 1 - \left(\frac{1}{64} + \frac{6}{64}\right) = \frac{57}{64} \]
(ii) \[ P(X < 4) = \sum_{x=0}^{3} \binom{6}{x} \left(\frac{1}{2}\right)^x \]

\[ = \frac{1}{64} + \frac{6}{64} + \frac{15}{64} + \frac{20}{64} = \frac{42}{64} = \frac{21}{32} \]

8.5. (a) Let \( p \) denote the probability of getting caught copying in the exam. Then \( p = 0.2, q = 0.8 \) and \( n = 3 \). Since the attempts are independent, therefore

\[ P(\text{not getting caught}) = q^3 = (0.8)^3 = 0.512 \]

(b) Let \( X \) denote the number of voters who prefer candidate \( A \) and \( p \), the probability that a voter prefers \( A \).

Then \( X = 7, p = 0.60 \) and \( n = 12 \). Hence

\[ P(X = 7) = \binom{12}{7} (0.60)^7 (0.40)^{12-7} \]

\[ = (792) (0.0279936) (0.01024) = 0.2270 \]

(c) Let \( X \) denote the number of patients recovering from a delicate heart operation, and \( p \), the probability that a patient recovers. Then \( p = 0.9, n = 7 \) and \( X = 5 \).

Thus \[ P(X = 5) = \binom{7}{5} (0.9)^5 (0.1)^2 \]

\[ = (21) (0.59049) (0.01) = 0.1240 \]

8.6 (a) Let \( X \) denote the number of workmen, and \( p \), the probability that a workman catches the disease. Then

\[ p = \frac{20}{100} = \frac{1}{5} \), so that \( q = 1 - \frac{1}{5} = \frac{4}{5} \), and \( n = 6 \).

(i) We seek the probability that "not more than 2" will catch the disease, i.e. \( X \leq 2 \). Hence

\[ P(X \leq 2) = \sum_{x=0}^{2} \binom{6}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{6-x} \]

\[ = \binom{6}{0} \left(\frac{4}{5}\right)^6 + \binom{6}{1} \left(\frac{1}{5}\right) \left(\frac{4}{5}\right)^5 + \binom{6}{2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4 \]
(ii) We seek the probability that 4 workmen, 5 workmen or 6 workmen will catch the disease. Hence

\[
P(X \geq 4) = \sum_{x=4}^{6} \binom{6}{x} \left( \frac{1}{5} \right)^x \left( \frac{4}{5} \right)^{6-x}
\]

\[
= \left( \frac{6}{4} \left( \frac{1}{5} \right)^4 \left( \frac{4}{5} \right)^2 \right) + \left( \frac{6}{5} \left( \frac{1}{5} \right)^5 \left( \frac{4}{5} \right)^1 \right) + \left( \frac{1}{5} \right)^6
\]

\[
= \frac{240}{15625} + \frac{24}{15625} + \frac{1}{15625} = \frac{265}{15625} = \frac{53}{3125}
\]

(b) Here \( p = \frac{12}{30} = 0.4 \) so that \( q = 0.6 \). Thus

(i) \( P(1\text{st} \ 3 \text{ days fine and the remaining} \ 4 \text{ days wet}) \)

\[
q^3 p^4 = (0.6)^3 (0.4)^4
\]

\[
= (0.216) (0.0256) = 0.0055.
\]

(ii) \( P(\text{rain falls on just three days}) = \binom{7}{3} p^3 q^4 \)

\[
= 35 (0.4)^3 (0.6)^4
\]

\[
= 35 (0.064) (0.1296)
\]

\[
= 0.2903
\]

8.7. We observe that

(i) there are two possible outcomes, \( i.e. \) a man will be alive or will not be alive.

(ii) the probability of being alive for each man is \( \frac{2}{3} \),

(iii) the men will remain alive independently, and

(iv) there are 5 men.

Therefore the binomial distribution with \( n=5 \) and \( p = \frac{2}{3} \) is appropriate to get the desired probabilities.
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Let \( X \) denote the number of people who will be alive in 30 years hence. Then we seek (i) \( P(X=5) \), (ii) \( P(X\geq3) \), (iii) \( P(X=2) \) and (iv) \( P(X\leq1) \). Hence

(i) \[ P(X=5) = \binom{5}{5} \left(\frac{2}{3}\right)^{5} \left(\frac{1}{3}\right)^{0} = \frac{32}{243} = 0.1317, \]

(ii) \[ P(X\geq3) = \sum_{x=3}^{5} \binom{5}{x} \left(\frac{2}{3}\right)^{x} \left(\frac{1}{3}\right)^{5-x} \]
\[ = \binom{5}{3} \left(\frac{2}{3}\right)^{3} \left(\frac{1}{3}\right)^{2} + \binom{5}{4} \left(\frac{2}{3}\right)^{4} \left(\frac{1}{3}\right)^{1} + \binom{5}{5} \left(\frac{2}{3}\right)^{5} \]
\[ = \frac{80}{243} + \frac{80}{243} + \frac{32}{243} = \frac{192}{243} = 0.7901; \]

(iii) \[ P(X=2) = \binom{5}{2} \left(\frac{2}{3}\right)^{2} \left(\frac{1}{3}\right)^{3} = \frac{40}{243} = 0.1646; \text{ and} \]

(iv) \[ P(X\leq1) = \sum_{x=0}^{1} \binom{5}{x} \left(\frac{2}{3}\right)^{x} \left(\frac{1}{3}\right)^{5-x} \]
\[ = \binom{5}{0} \left(\frac{2}{3}\right)^{0} \left(\frac{1}{3}\right)^{5} + \binom{5}{1} \left(\frac{2}{3}\right)^{1} \left(\frac{1}{3}\right)^{4} \]
\[ = \frac{1}{243} + \frac{10}{243} = \frac{11}{243} = 0.0453. \]

8.8. (a) The probability of a correct answer is \( \frac{1}{4} = 0.25 \), and \( n = 15 \).

Let \( X \) denote the number of correct answers. Then

\[ P(5\leq X\leq10) = \sum_{x=5}^{10} \binom{15}{x} (0.25)^{x} (0.75)^{15-x} = \sum_{x=5}^{10} b(x; 15, 0.25) \]
\[ = \sum_{x=0}^{4} b(x; 15, 0.25) = 0.9999 - 0.6865 \text{ (From binomial tables),} \]
\[ = 0.3134. \]
(b) Let $p$ denote the probability that a spot light is red when the commuter gets to it. Then $p = 0.2$, $q = 1 - 0.2 = 0.8$ and $n = 10$.

Let $X$ be a r.v. that denotes the number of times, the commuter must stop for a red light on her way to work. Then $X$ is $b(x; 10, 0.2)$. To evaluate $P(X = 0)$ and $P(X \leq 5)$, we do the following calculations.

Now $P(X = x) = \binom{10}{x} (0.2)^x (0.8)^{10-x}$, so

$P(X = 0) = \binom{10}{0} (0.2)^0 (0.8)^{10} = 0.107374$ and

$P(X \leq 5) = \sum_{x=0}^{5} \binom{10}{x} (0.2)^x (0.8)^{10-x}$

$= (0.8)^{10} + \binom{10}{1}(0.2)(0.8)^9 + \binom{10}{2}(0.2)^2(0.8)^8 + \binom{10}{3}(0.2)^3(0.8)^7 + \binom{10}{4}(0.2)^4(0.8)^6 + \binom{10}{5}(0.2)^5(0.8)^5$

$= 0.107374 + 0.268345 + 0.301990 + 0.201327 + 0.088080 + 0.026424 = 0.9936$.

8.9. (a) The successive terms of the binomial distribution $600(0.3 + 0.7)^6$ are calculated below:

<table>
<thead>
<tr>
<th>$X$</th>
<th>Probability</th>
<th>$f(x)$</th>
<th>$600 \times f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q^6 = (0.3)^6$</td>
<td>$0.000729$</td>
<td>$0.4374$</td>
</tr>
<tr>
<td>1</td>
<td>$6C_1 q^5p = 6(0.3)^5 (0.7)$</td>
<td>$0.010206$</td>
<td>$6.1236$</td>
</tr>
<tr>
<td>2</td>
<td>$6C_2 q^4p^2 = 15(0.3)^4 (0.7)^2$</td>
<td>$0.059535$</td>
<td>$35.7210$</td>
</tr>
<tr>
<td>3</td>
<td>$6C_3 q^3p^3 = 20(0.3)^3 (0.7)^3$</td>
<td>$0.185220$</td>
<td>$111.1320$</td>
</tr>
<tr>
<td>4</td>
<td>$6C_4 q^2p^4 = 15(0.3)^2 (0.7)^4$</td>
<td>$0.324135$</td>
<td>$194.4810$</td>
</tr>
<tr>
<td>5</td>
<td>$6C_5 qp^5 = 6(0.3) (0.7)^5$</td>
<td>$0.302526$</td>
<td>$181.5156$</td>
</tr>
<tr>
<td>6</td>
<td>$p^6 = (0.7)^6$</td>
<td>$0.117649$</td>
<td>$70.5894$</td>
</tr>
</tbody>
</table>
(b) Let \( p \) denote the probability of getting 4, 5 or 6 with one die.

Then \( p = \frac{3}{6} = \frac{1}{2} \), so that \( q = \frac{1}{2} \).

We seek the expected frequencies of getting 0, 1, 2, ..., 5 successes with 5 dice in tossing 96 times, which are the successive terms in the binomial expansion of

\[
96 \left( \frac{1}{2} + \frac{1}{2} \right)^5
\]

Hence the expected frequencies are:

\[
96 \left( \frac{1}{2} \right)^5; 96 \left( \frac{5}{1} \right) \left( \frac{1}{2} \right)^5; 96 \left( \frac{5}{2} \right) \left( \frac{1}{2} \right)^5, 96 \left( \frac{5}{3} \right) \left( \frac{1}{2} \right)^5, 96 \left( \frac{5}{4} \right) \left( \frac{1}{2} \right)^5 \text{ and } 96 \cdot \left( \frac{1}{2} \right)^5 \text{ or 3, 15, 30, 30, 15 and 3}
\]

8.10. (a) Let \( p \) denote the probability of getting a 5 or a 6 with one die.

Then \( p = \frac{2}{6} = \frac{1}{3} \), so that \( q = 1 - \frac{1}{3} = \frac{2}{3} \).

As the dice are in sets of 8, the binomial distribution is

\[
N \left( \frac{2}{3} + \frac{1}{3} \right)^8
\]

Thus the expected number of 3 successes

\[
= N \cdot \binom{8}{3} \left( \frac{1}{3} \right)^3 \left( \frac{2}{3} \right)^5 = N \cdot \frac{1792}{6561}
\]

Hence the proportion of the sets in which 3 successes are expected

\[
= N \cdot \frac{1792}{6561} \div N = \frac{1792}{6561} \times 100% = 27.31%
\]
(b) Let \( p \) denote the probability of getting an even number.

Then the expected number of getting 5 even numbers in 10 throws = \( N \cdot \binom{10}{5} p^5 q^5 \), where \( N \) is the no. of sets and \( q = 1 - p \).

Similarly, the expected number of getting 4 even numbers in 10 throws = \( N \cdot \binom{10}{4} p^4 q^6 \).

Now, by the question, we have

\[
N \cdot \binom{10}{5} p^5 q^5 = 2N \cdot \binom{10}{4} p^4 q^6
\]

Or

\[
\frac{10!}{5! \cdot 5!} p^5 q^5 = 2 \cdot \frac{10!}{4! \cdot 6!} p^4 q^6
\]

Or

\[
p = \frac{q}{3} \quad \text{Or} \quad 3p = 5(1 - p)
\]

Or

\[p = \frac{5}{8} \quad \text{and hence} \quad q = \frac{3}{8}.
\]

Then the binomial distribution becomes

\[10,000 \left( \frac{3}{8} + \frac{5}{8} \right)^{10}\]

Hence the expected number of getting no even number

\[= 10,000 \left( \frac{3}{8} \right)^{10} = \frac{36905625}{67108864} = 1 \text{ approximately.}
\]

8.11. Let \( p \) be the probability of getting a 6 with one die. Then \( p = \frac{1}{6} \) and \( q = 1 - p = \frac{5}{6} \).

Let \( X \) denote the number of sixes when 4 dice are thrown. Then the theoretical frequencies of 0, 1, 2, 3, and 4 sixes are the successive terms of the binomial distribution \( 108 \left( \frac{5}{6} + \frac{1}{6} \right)^4 \), which are calculated as follows:
### Chapter # 8
**Discrete Probability Distribution**

<table>
<thead>
<tr>
<th>x</th>
<th>Probability</th>
<th>$f(x)$</th>
<th>$108 \times f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q^4 = \left(\frac{5}{6}\right)^4$</td>
<td>$\frac{625}{1296}$</td>
<td>52.08</td>
</tr>
<tr>
<td>1</td>
<td>$\binom{4}{1} q^3p = 4 \cdot \left(\frac{5}{6}\right)^3 \cdot \left(\frac{1}{6}\right)$</td>
<td>$\frac{500}{1296}$</td>
<td>41.67</td>
</tr>
<tr>
<td>2</td>
<td>$\binom{4}{2} q^2p^2 = 6 \cdot \left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{6}\right)^2$</td>
<td>$\frac{150}{1296}$</td>
<td>12.50</td>
</tr>
<tr>
<td>3</td>
<td>$\binom{4}{3} qp^3 = 4 \cdot \left(\frac{5}{6}\right) \cdot \left(\frac{1}{6}\right)^3$</td>
<td>$\frac{20}{1296}$</td>
<td>1.67</td>
</tr>
<tr>
<td>4</td>
<td>$p^4 = \left(\frac{1}{6}\right)^4$</td>
<td>$\frac{1}{1296}$</td>
<td>0.08</td>
</tr>
</tbody>
</table>

The mean number of sixes in a single throw $n p = 4 \times \frac{1}{6} = \frac{2}{3}$.

### 8.12. (a) Calculation of the mean and standard deviation of the binomial $(q+p)^3$.

<table>
<thead>
<tr>
<th>x</th>
<th>$f(x)$</th>
<th>$x \cdot f(x)$</th>
<th>$x^2 \cdot f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q^3$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$3q^2p$</td>
<td>$3q^2p$</td>
<td>$3q^2p$</td>
</tr>
<tr>
<td>2</td>
<td>$3qp^2$</td>
<td>$6qp^2$</td>
<td>$12qp^2$</td>
</tr>
<tr>
<td>3</td>
<td>$p^3$</td>
<td>$3p^3$</td>
<td>$9p^3$</td>
</tr>
<tr>
<td>$\sum$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now $\sum x \cdot f(x) = 3q^2p + 6qp^2 + 3p^3$

$= 3p \left[ q^2 + 2qp + p^2 \right] = 3p(q + p)^2 = 3p$.

$\sum x^2 \cdot f(x) = 3q^2p + 12qp^2 + 9p^3$

$= 3p \left[ q^2 + 4qp + 3p^2 \right]$

$= 3p \left[ (q^2 + 2qp + p^2) + (2qp + 2p^2) \right]$

$= 3p \left[ (q + p)^2 + 2p(q + p) \right]$

$= 3p \left[ 1 + 2p \right] = 3p + 6p^2$

Hence $\mu = \sum x \cdot f(x) = 3p$, and $\sigma = \sqrt{\sum x^2 \cdot f(x) - (\mu)^2} = \sqrt{3p + 6p^2 - (3p)^2} = \sqrt{7p - 3p^2}$.

$(n = \sum f(x) = 1)$
\[ \sigma = \sqrt{\sum x^2 \cdot f(x) - [\sum x \cdot f(x)]^2} \]
\[ = \sqrt{3p + 6p^2 - (3p)^2} \]
\[ = \sqrt{3p - 3p^2} = \sqrt{3p (1 - p)} = \sqrt{3pq} \]

(c) As the mean and variance of the binomial \((q+p)^n\) are \(np\) and \(npq\), therefore \(np = 3\) and \(npq = 2\).

Dividing, we get
\[ \frac{npq}{np} = \frac{2}{3} \quad \text{or} \quad q = \frac{2}{3} \cdot \]

\[ \therefore \quad p = 1 - q = \frac{1}{3}, \quad \text{and} \quad n \left(\frac{1}{3}\right) = 3, \quad \text{gives} \quad n = 9. \]

Thus the binomial distribution is \(\left(\frac{2}{3} + \frac{1}{3}\right)^9\).

Hence \(P(X=7) = \binom{9}{7} \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right)^2 = \frac{16}{2187} = 0.0073\).

8.13 (a) The mean and the standard deviation of the binomial \((q+p)^n\) are \(np\) and \(\sqrt{npq}\). Then
\[ np = 36 \quad \text{and} \quad \sqrt{npq} = 4.8 \quad \text{or} \quad npq = (4.8)^2 \]

Dividing, we get
\[ \frac{npq}{np} = \frac{(4.8)^2}{36} \quad \text{or} \quad q = 0.64 \]

\[ \therefore \quad p = 1 - q = 0.36 \]

Putting \(p = 0.36\) in \(np = 36\), we get
\[ n = \frac{36}{0.36} = 100. \]

(b) If \(X\) is a binomial r.v. with parameters \(n\) and \(p\), then mean = \(np\) and variance = \(npq\).

Now mean = 12.38 so \(np = 12.38\), and

Variance = 8.64 so \(npq = 8.64\).

Substituting for \(np\) in the second equation, we get
(12.38) \( q = 8.64 \) or \( q = \frac{8.64}{12.38} = 0.6979 \)

so \( p = 1 - 0.6979 = 0.3021 \), and
\( n(0.3021) = 12.38 \) gives \( n = 41 \).

(c) Let the binomial distribution be \((q+p)^n; q + p = 1\).

Then \( np = 5 \) and \( \sqrt{npq} = 3 \) or \( npq = 9 \)

Dividing, we get
\[
\frac{npq}{np} = \frac{9}{5} \text{ or } q = 1.8, \text{ which is greater than unity.}
\]

Hence it is not possible to have a binomial distribution with mean = 5 and s.d. = 3.

8.17. (a) The binomial distribution of the r.v. \( X \) is given by
\[
P(X=x) = \binom{25}{x} (0.2)^x (0.8)^{25-x}.
\]

Now \( \mu = np = 25 \times 0.2 = 5 \), and
\( \sigma = \sqrt{npq} = \sqrt{25 \times 0.2 \times 0.8} = 2 \).

Hence \( P(X<\mu-2\sigma) = P(X<1) \quad (\because \mu-2\sigma = 5-2(2) = 1) \)
\[
= P(X=0) = \binom{25}{0} (0.2)^0 (0.8)^{25} = 0.00378.
\]

(b) To find the median and mode of the binomial
\[
\binom{10}{x} \left( \frac{1}{2} \right)^x \left( \frac{1}{2} \right)^{10-x},
\]
we expand it. Therefore, expanding the binomial, we get

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
</tr>
<tr>
<td>( F )</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
<td>1024</td>
</tr>
</tbody>
</table>

Since the binomial distribution is a discrete distribution, therefore
\[
\text{Median} = \text{a value at or below which } 50\% \text{ probabilities lie}
\]
\[
= 5; \text{ and}
\]
Mode = a value that corresponds to maximum probability

= 5.

8.18. (a) To find the probability of obtaining a head, we need to calculate the mean of the frequency distribution. Thus

\[ x = \frac{\sum fx}{\sum f} = \frac{(0)(12) + (1)(50) + (2)(151) + (3)(200) + (4)(87)}{500} \]

\[ = \frac{1300}{500} = 2.6 \]

Let \( X \) denote the number of heads obtained in 4 tosses. Then \( X \) is a binomial r.v. with \( n=4 \), and mean \( \mu = np = 4p \).

Now \( 4p = 2.6 \) or \( p = 0.65 \)

Thus the probability that the coin will show heads is 0.65.

(b) Now \( X \) is \( b(x; 4, 0.65) \) and therefore

\[ P(X=x) = \binom{4}{x} (0.65)^x (0.35)^{4-x} \] for \( x = 0,1,2,3,4. \)

The theoretical binomial probabilities and frequencies are computed as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>Probability</th>
<th>( = f(x) )</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \binom{4}{0} (0.65)^0 (0.35)^4 ) = 0.015006</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( \binom{4}{1} (0.65)^1 (0.35)^3 ) = 0.111475</td>
<td>56</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \binom{4}{2} (0.65)^2 (0.35)^2 ) = 0.310538</td>
<td>155</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( \binom{4}{3} (0.65)^3 (0.35)^1 ) = 0.384475</td>
<td>192</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( \binom{4}{4} (0.65)^4 (0.35)^0 ) = 0.178506</td>
<td>89</td>
<td></td>
</tr>
</tbody>
</table>
8.19. We first calculate the mean of the given distribution and equate it to \( np \) to find the value of \( p \).

Now \[
x = \frac{\sum f_i x_i}{\sum f_i} = \frac{0 + 70 + 122 + 75 + 28 + 5 + 0}{200} = \frac{300}{200} = 1.5.
\]

Thus \( 6p = 1.5 \) or \( p = 0.25 \) and hence \( q = 0.75 \).

The fitted binomial distribution is

\[
b(x; 6, 0.25) = \binom{6}{x} (0.25)^x (0.75)^{6-x}
\]

The theoretical binomial probabilities and frequencies are computed as below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>Probability</th>
<th>Theoretical Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \binom{6}{0} (0.75)^6 )</td>
<td>0.177978</td>
</tr>
<tr>
<td>1</td>
<td>( \binom{6}{1} (0.25) (0.75)^5 )</td>
<td>0.355957</td>
</tr>
<tr>
<td>2</td>
<td>( \binom{6}{2} (0.25)^2 (0.75)^4 )</td>
<td>0.296631</td>
</tr>
<tr>
<td>3</td>
<td>( \binom{6}{3} (0.25)^3 (0.75)^3 )</td>
<td>0.131836</td>
</tr>
<tr>
<td>4</td>
<td>( \binom{6}{4} (0.25)^4 (0.75)^2 )</td>
<td>0.032959</td>
</tr>
<tr>
<td>5</td>
<td>( \binom{6}{5} (0.25)^5 (0.75) )</td>
<td>0.004394</td>
</tr>
<tr>
<td>6</td>
<td>( \binom{6}{6} (0.25)^6 )</td>
<td>0.000244</td>
</tr>
<tr>
<td>( \sum )</td>
<td>= 0.999999</td>
<td>200.01</td>
</tr>
</tbody>
</table>

8.20. Fitting of the Binomial Distribution.

We first calculate the mean of the given distribution and equate it to \( np \) to find the value of \( p \).
CHAPTER # 8
DISCRETE PROBABILITY DISTRIBUTION

Now \[ x = \frac{\sum f_i x_i}{\sum f_i} = \frac{0 + 62 + 92 + 30 + 8}{150} = \frac{192}{150} = 1.28. \]

Thus \( 4p = 1.28 \) or \( p = 0.32 \) and hence \( q = 0.68 \).

The fitted binomial distribution is

\[ b(x; 4, 0.32) = \binom{4}{x} (0.32)^x (0.68)^{4-x} \]

The expected frequencies are computed as below:

<table>
<thead>
<tr>
<th>x</th>
<th>Probability</th>
<th>Expected frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>((0.68)^4)</td>
<td>0.213814</td>
</tr>
<tr>
<td>1</td>
<td>(\binom{4}{1} (0.32)(0.68)^3)</td>
<td>0.402473</td>
</tr>
<tr>
<td>2</td>
<td>(\binom{4}{2} (0.32)^2 (0.68)^2)</td>
<td>0.284099</td>
</tr>
<tr>
<td>3</td>
<td>(\binom{4}{3} (0.32)^3 (0.68))</td>
<td>0.089129</td>
</tr>
<tr>
<td>4</td>
<td>((0.32)^4)</td>
<td>0.010486</td>
</tr>
<tr>
<td>Σ</td>
<td></td>
<td>1.000001</td>
</tr>
</tbody>
</table>

8.21. Calculation of the mean and standard deviation.

<table>
<thead>
<tr>
<th>x</th>
<th>f</th>
<th>fx</th>
<th>f(x^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>24</td>
<td>72</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>64</td>
<td>256</td>
</tr>
<tr>
<td>5</td>
<td>28</td>
<td>140</td>
<td>700</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
<td>108</td>
<td>648</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>91</td>
<td>637</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>72</td>
<td>576</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>36</td>
<td>324</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Total 100 542 3226
\[ \bar{x} = \frac{\sum fx}{\sum f} = \frac{542}{100} = 5.42, \text{ and} \]

\[ s = \sqrt{\frac{\sum f x^2}{\sum f} - \left( \frac{\sum f x}{\sum f} \right)^2} = \sqrt{\frac{3226}{100} - \left( \frac{542}{100} \right)^2} \]

\[ = \sqrt{32.26 - 29.3764} = \sqrt{2.8836} = 1.70. \]

Now putting the mean equal to np, we get

10p = 5.42 so that p = 0.542 and q = 0.458.

Thus the fitted binomial distribution is

\[ b(x; 10, 0.542) = \binom{10}{x} (0.542)^x \cdot (0.458)^{10-x} \]

The expected frequencies are computed as below:

<table>
<thead>
<tr>
<th>x</th>
<th>Probability</th>
<th>Expected frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0.458)^10</td>
<td>0.000406</td>
</tr>
<tr>
<td>1</td>
<td>(10) (1)</td>
<td>(0.542)(0.458)^9)</td>
</tr>
<tr>
<td>2</td>
<td>(10) (2)</td>
<td>(0.542)^2 (0.458)^8)</td>
</tr>
<tr>
<td>3</td>
<td>(10) (3)</td>
<td>(0.542)^3 (0.458)^7)</td>
</tr>
<tr>
<td>4</td>
<td>(10) (4)</td>
<td>(0.542)^4 (0.458)^6)</td>
</tr>
<tr>
<td>5</td>
<td>(10) (5)</td>
<td>(0.542)^5 (0.458)^5)</td>
</tr>
<tr>
<td>6</td>
<td>(10) (6)</td>
<td>(0.542)^6 (0.458)^4)</td>
</tr>
<tr>
<td>7</td>
<td>(10) (7)</td>
<td>(0.542)^7 (0.458)^3)</td>
</tr>
<tr>
<td>8</td>
<td>(10) (8)</td>
<td>(0.542)^8 (0.458)^2)</td>
</tr>
<tr>
<td>9</td>
<td>(10) (9)</td>
<td>(0.542)^9 (0.458))</td>
</tr>
<tr>
<td>10</td>
<td>(0.542)^10</td>
<td>0.002188</td>
</tr>
<tr>
<td>(\sum)</td>
<td></td>
<td>= 1.000192</td>
</tr>
</tbody>
</table>
Mean and variance of the expected distribution are

\[
\mu = np = 10(0.542) = 5.42
\]

\[
\sigma = \sqrt{npq} = \sqrt{10(0.542)(0.458)}
\]

\[
= \sqrt{2.48236} = 1.57
\]

8.22. For a symmetrical binomial distribution, we have

\[
p = q = \frac{1}{2}
\]

Then the symmetrical binomial distribution of degree \(n\) and observations, \(N\), is \(N\left(\frac{1}{2} + \frac{1}{2}\right)^n\).

Let \(T_r\) and \(T_{r+1}\) denote the \(r\)th and \((r+1)\)th terms. Then

\[
T_r = N \cdot \binom{n}{r-1} \left(\frac{1}{2}\right)^{r-1} \left(\frac{1}{2}\right)^{n-r+1}
\]

\[
= N \cdot \binom{n}{r-1} \left(\frac{1}{2}\right)^n, \text{ and}
\]

\[
T_{r+1} = N \cdot \binom{n}{r} \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{n-r}
\]

Superposing in such a way that the \(r\)th term of one coincides with the \((r+1)\)th term of the other, we get

\[
T_r + T_{r+1} = N \cdot \binom{n}{r-1} \left(\frac{1}{2}\right)^r + N \cdot \binom{n}{r} \left(\frac{1}{2}\right)^n
\]

\[
= N \left[ \binom{n}{r-1} + \binom{n}{r} \right] \left(\frac{1}{2}\right)^n
\]

\[
= N \left(\binom{n+1}{r} \left(\frac{1}{2}\right)^n \right) + 2N \left(\binom{n+1}{r} \left(\frac{1}{2}\right)^{n+1} \right)
\]

\[
= 2N \cdot \binom{n+1}{r} \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{n+1-r}
\]
(r + 1)th term of the binomial distribution

\[ 2N \cdot \left( \frac{1}{2} + \frac{1}{2} \right)^{n+1} \]

, which is symmetrical binomial of degree \((n + 1)\).

8.23. (b) Given that

\[ M_0(t) = \left( \frac{1}{4} + \frac{3}{4} e^t \right)^{12} \]

Then

\[ E(X) = \left[ \frac{d}{dt} \left( \frac{1}{4} + \frac{3}{4} e^t \right)^{12} \right]_{t=0} \]

\[ = \left[ 12 \left( \frac{3}{4} e^t \right) \left( \frac{1}{4} + \frac{3}{4} e^t \right)^{11} \right]_{t=0} = 12 \left( \frac{3}{4} \right) (1) = 9; \]

\[ E(X^2) = \left[ \frac{d^2}{dt^2} \left( \frac{1}{4} + \frac{3}{4} e^t \right)^{12} \right]_{t=0} \]

\[ = \left[ 12 \left( \frac{3}{4} e^t \right) \left( \frac{1}{4} + \frac{3}{4} e^t \right)^{11} \right]_{t=0} \]

\[ + \left[ 12 (11) \left( \frac{3}{4} \right)^2 e^{2t} \left( \frac{1}{4} + \frac{3}{4} e^t \right)^{10} \right]_{t=0} \]

\[ = 12 \left( \frac{3}{4} \right) (1) + 12(11) \left( \frac{3}{4} \right)^2 (1) = 9 + \frac{297}{4} = \frac{333}{4}. \]

\[ \therefore \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{333}{4} - (9)^2 = \frac{333 - 324}{4} = \frac{9}{4} = 2.25. \]

Alternatively. Comparing \( \left( \frac{1}{4} + \frac{3}{4} e^t \right)^{12} \) with \((q + pe^t)^n\), we find that \( p = \frac{3}{4}, q = \frac{1}{4} \) and \( n = 12 \).

Therefore \( E(X) = np = 12 \times \frac{3}{4} = 9 \); and

\[ \text{Var}(X) = npq = 12 \times \frac{3}{4} \times \frac{1}{4} = \frac{9}{4} = 2.25. \]

Now \[ P(X \geq 10) = \sum_{x=10}^{12} \binom{12}{x} \left( \frac{3}{4} \right)^x \left( \frac{1}{4} \right)^{12-x} \]

\[ = \binom{12}{10} \left( \frac{3}{4} \right)^{10} \left( \frac{1}{4} \right)^2 + \binom{12}{11} \left( \frac{3}{4} \right)^{11} \left( \frac{1}{4} \right) + \left( \frac{3}{4} \right)^{12} \]
\[
(66) \left( \frac{59049}{1048576} \right) \left( \frac{1}{16} \right) + 12 \left( \frac{177147}{4194304} \right) \left( \frac{1}{4} \right) + \left( \frac{531441}{16777216} \right)
= 0.232293 + 0.126705 + 0.031676
= 0.3907.
\]

8.24. The m.g.f. of the binomial distribution \((q+p)^n\) is

\[ M_0(t) = (q + pe^t)^n, \] and

the cumulant generating function is

\[ \kappa(t) = n \log (q + pe^t), \]

therefore \(r\)th order cumulant is given by

\[ \kappa_r = n \left[ \frac{d^r}{dt^r} \log (q + pe^t) \right] \]

Differentiating with respect to \(p\), we get

\[ \frac{d\kappa_r}{dp} = n \left[ \frac{d^r}{dt^r} \left( \frac{-1+e^t}{q+pe^t} \right) \right] \]

\[ \therefore q + pe^t = 1 - p + pe^t. \]

Now
\[ pq \frac{d\kappa_r}{dp} = npq \left[ \frac{d^r}{dt^r} \left( \frac{-1+e^t}{q+pe^t} \right) \right] \]

\[ \therefore \frac{d}{dp} = -1 + e^t \]

Again
\[ \kappa_{r+1} = n \left[ \frac{d^{r+1}}{dt^{r+1}} \log (q + pe^t) \right] \]

\[ = n \left[ \frac{d^r}{dt^r} \cdot \frac{d}{dt} \log (q + pe^t) \right] = np \left[ \frac{d^r}{dt^r} \left( \frac{e^t}{q + pe^t} \right) \right] \]

\[ \therefore \kappa_{r+1} - pq \frac{d\kappa_r}{dp} = np \left[ \frac{d^r}{dt^r} \left( \frac{e^t}{q + pe^t} - \frac{q(-1+e^t)}{q + pe^t} \right) \right]_{t=0}
= np \left[ \frac{d^r}{dt^r} \left( \frac{q + pe^t}{q + pe^t} \right) \right]_{t=0} = np \left[ \frac{d^r}{dt^r} (1) \right]_{t=0} = 0, \]

so that \( \kappa_{r+1} = pq \frac{d\kappa_r}{dp} \).

Putting \( r = 1 \), we get \( \kappa_2 = pq \frac{d\kappa_1}{dp} \), where \( \kappa_1 = np \)

\[ = npq \]
Putting $r = 2$, we get $\kappa_3 = npq \frac{dK_2}{dp} = npq \frac{d}{dp} (qp)$

$$= npq (1 - 2p) = npq (q - p)$$

$$= n(p - 3p^2 + 2p^3)$$

Putting $r = 3$, we get $\kappa_4 = npq \frac{dK_3}{dp} = npq \frac{d}{dp} (p - 3p^2 + 2p^3)$

$$= npq (1 - 6p + 6p^2)$$

$$= npq (1 - 6p).$$

8.26. (b) Let $X$ denote the number of white beads. Then the probability that $X$ assumes a value $x$, is

$$P(X=x) = \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}}$$

for $x$ such that $0 \leq x \leq n$ and $0 \leq x \leq k$.

where $k =$ number of white beads, which is 4;

$n =$ number of beads drawn, which is 5,

$N =$ total number of beads in the bowl which is 11, and

$x = 0,1,2,3,$ and 4. (Here possible values of $x$ are equal to $k$)

Now we calculate the probabilities for various values of $x$ as below:

$$P(X=0) = \frac{\binom{4}{0} \binom{11-4}{5-0}}{\binom{11}{5}} = \frac{21}{462},$$

$$P(X=1) = \frac{\binom{4}{1} \binom{7}{4}}{\binom{11}{5}} = \frac{140}{462},$$

$$P(X=2) = \frac{\binom{4}{2} \binom{7}{3}}{\binom{11}{5}} = \frac{210}{462},$$

$$P(X=3) = \frac{\binom{4}{3} \binom{7}{2}}{\binom{11}{5}} = \frac{84}{462},$$

$$P(X=4) = \frac{\binom{4}{4} \binom{7}{1}}{\binom{11}{5}} = \frac{7}{462},$$
Hence the desired probability distribution is

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X=x)$</td>
<td>21</td>
<td>140</td>
<td>210</td>
<td>84</td>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

Now $\sum xf(x) = (0)\left(\frac{21}{462}\right) + 1\left(\frac{140}{462}\right) + 2\left(\frac{210}{462}\right) + 3\left(\frac{84}{462}\right) + 4\left(\frac{7}{462}\right)$

$$= 0 + \frac{140}{462} + \frac{420}{462} + \frac{252}{462} + \frac{28}{462} = \frac{840}{462}, \text{ and }$$

$$\sum x^2 f(x) = 0\left(\frac{21}{462}\right) + 1^2\left(\frac{140}{462}\right) + 2^2\left(\frac{210}{462}\right) + 3^2\left(\frac{84}{462}\right) + 4^2\left(\frac{7}{462}\right)$$

$$= 0 + \frac{140}{462} + \frac{840}{462} + \frac{756}{462} + \frac{112}{462} = \frac{1848}{462}.$$  

Hence $\text{Mean} = \sum x_i f(x_i) = \frac{840}{462} = 1.82$, and

$$\text{Variance} = \sum x_i^2 f(x_i) - \left(\sum x_i f(x_i)\right)^2 = \frac{1848}{462} - \left(\frac{840}{462}\right)^2$$

$$= 4 - \frac{3.31}{462} = 0.69.$$  

The formulas for the mean and variance of the distribution are:

$$\mu = np \quad \text{and} \quad \sigma^2 = npq \frac{N-n}{N-1}, \text{ where } p = \frac{k}{N} \quad \text{and} \quad q = \frac{N-k}{N}$$

Here $p = \frac{4}{11}$ and $q = \frac{7}{11}$. Therefore

$$\mu = 5\left(\frac{4}{11}\right) = \frac{20}{11} = 1.82, \text{ and}$$

$$\sigma^2 = 5\left(\frac{4}{11}\right)\left(\frac{7}{11}\right)\frac{11 - 5}{11 - 1} = \frac{84}{121} = 0.69.$$  

8.27. (a) Here $N = 4 \text{ men} + 2 \text{ women} = 6 \text{ persons},$

$n = 3 \text{ persons to be selected},$

$k = 4, \text{ and } x = 0, 1, 2, 3.$
Hence the Hypergeometric distribution is

\[ h(x; 6, 3, 4) = \binom{4}{x} \binom{2}{3-x} \binom{6}{3} \]

When \( x=0 \), \( h(0; 6, 3, 4) = 0 \), (the event is impossible)

When \( x = 1 \), the probability is

\[ h(1; 6, 3, 4) = \frac{\binom{4}{1} \binom{2}{2}}{\binom{6}{3}} = \frac{4}{20} \]

When \( x = 2 \), the probability is

\[ h(2; 6, 3, 4) = \frac{\binom{4}{2} \binom{2}{1}}{\binom{6}{3}} = \frac{12}{20} \]

When \( x = 3 \), the probability is

\[ h(3; 6, 3, 4) = \frac{\binom{4}{3} \binom{2}{0}}{\binom{6}{3}} = \frac{4}{20} \]

(b) Here 6 bulbs can be selected in \( \binom{8}{6} \) ways,

4 tulip bulbs can be selected in \( \binom{4}{4} \) ways, and

2 daffodil bulbs can be selected in \( \binom{4}{2} \) ways.

\[ \therefore \text{ the total number of ways to select 2 daffodil bulbs and 4 tulip bulbs is } \binom{4}{2} \binom{4}{4} \text{ ways.} \]

Hence the required probability \[ = \frac{\binom{4}{2} \binom{4}{4}}{\binom{8}{6}} = \frac{4 \times 3 \times 2}{2 \times 8 \times 7} = \frac{3}{14} \]
Or Applying Hypergeometric distribution, we identify

\[ N = 8, \, n = 6, \, k = 4, \, x = 2. \]

\[ h(2; 8, 6, 4) = \frac{\binom{4}{2} \cdot \binom{8-4}{6-2}}{\binom{8}{6}} = \frac{3}{14}. \]

8.28. Here \( N = 10 \) cans to draw from,

\( n = 5 \), the number of cans to be drawn.

\( k = 5 \) of the 10 cans are tomatoes.

Thus the Hypergeometric distribution is

\[ h(x; 10, 5, 5) = \frac{\binom{5}{x} \cdot \binom{5}{5-x}}{\binom{10}{5}}. \]

Hence the probability that all contain tomatoes, \( i.e. \, x=5 \) is

\[ h(5; 10, 5, 5) = \frac{\binom{5}{5} \cdot \binom{5}{0}}{\binom{10}{5}} = 0.003968. \]

Let \( A \) denote the event that 3 or more cans contain tomatoes. Then

\[ P(A) = \sum_{x=3}^{5} \frac{\binom{5}{x} \cdot \binom{5}{5-x}}{\binom{10}{5}} \]

\[ = \left[ \frac{\binom{5}{3} \cdot \binom{5}{2} + \binom{5}{4} \cdot \binom{5}{1} + \binom{5}{5} \cdot \binom{5}{0}}{\binom{10}{5}} \right] \]

\[ = \frac{(100 + 25 + 1)}{252} = 0.5 \]

8.29. (a) Let \( X \) denote the number of income tax returns with illegitimate deductions. Then we have

\[ N = 20, \, n = 6, \, k = 8 \, \text{and} \, x = 3. \]

Hence using the hypergeometric distribution
\[ h(x; 20, 6, 8) = \frac{\binom{8}{x} \binom{12}{6-x}}{\binom{20}{6}}, \text{ we get the desired probability,} \]

\[ i.e. P(X=3), \text{ as} \]

\[ h(3; 20, 6, 8) = \frac{\binom{8}{3} \binom{12}{3}}{\binom{20}{6}} = \frac{12320}{38760} = \frac{308}{969} = 0.3179. \]

(b) Let \( X \) denote the number of narcotic tablets. Then we have

\[ k = 6, \ n = 3 \ \text{and} \ N = 6 + 9 = 15. \]

The hypergeometric distribution is then

\[ P(X=x) = h(x; 15, 3, 6) = \frac{\binom{6}{x} \binom{9}{3-x}}{\binom{15}{3}}. \]

The traveller will be arrested for illegal possession of narcotics if the customs official detects at least 1 narcotic tablet, \( i.e. \) if \( x = 1, 2, 3. \)

Hence the desired probability is

\[ \sum_{x=1}^{3} h(x; 15, 3, 6) = \sum_{x=1}^{3} \frac{\binom{6}{x} \binom{9}{3-x}}{\binom{15}{3}}. \]

\[ = \frac{\binom{6}{1} \binom{9}{2}}{455} + \frac{\binom{6}{2} \binom{9}{1}}{455} + \frac{\binom{6}{3} \binom{9}{0}}{455} \]

\[ = \frac{216}{455} + \frac{135}{455} + \frac{20}{455} = \frac{371}{455} = 0.82. \]

8.31. (a) When a sample is obtained under sampling without replacement, the hypergeometric distribution is used to find the desired probability. We are given that

\[ N = 150, \ n = 4, \ x = 3 \ \text{and} \ k = 20\% \text{ of} \ 150 = 30. \]
\[
P(X=3) = h(3; 150, 4, 30) = \frac{\binom{30}{3} \binom{120}{1}}{150^4} = 0.0240
\]

(b) When a sample is obtained under sampling with replacement from a finite population or sampling from infinite population, the binomial probability is appropriate to calculate the desired probability. We are given that \( n = 4, \, x = 3 \) and \( p = 0.20 \).

\[
\therefore \quad P(X=3) = b(3; 4, 0.20) = \binom{4}{3} (0.20)^3 (0.80) = 0.0256.
\]

(c) There is a slight difference because the sample size is small.

8.32. (c) Since \( X \) is a Poisson r.v. with \( \mu = 1.6 \), therefore

\[
P(X=0) = e^{-1.6} \frac{(1.6)^0}{0!} = 0.2019, \quad (\therefore \ e^{-1.6} = 0.2019)
\]

\[
P(X=1) = \frac{e^{-1.6} (1.6)^1}{1!} = 0.3230,
\]

\[
P(X=2) = \frac{e^{-1.6} (1.6)^2}{2!} = 0.2584, \text{ and}
\]

\[
P(X>2) = 1 - P(X \leq 2)
= 1 - [P(X=0) + P(X=1) + P(X=2)]
= 1 - 0.7833 = 0.2167.
\]

8.33. (a) The desired probabilities for \( x = 0, 1, 2, 3 \) and \( 4 \) are computed as below:

\[
P(X=0) = e^{-3} \cdot \frac{3^0}{0!} = e^{-3} = 0.0498,
\]

\[
P(X=1) = e^{-3} \cdot \frac{3^1}{1!} = (0.0498) \cdot (3) = 0.1494,
\]

\[
P(X=2) = e^{-3} \cdot \frac{3^2}{2!} = (0.1494) \cdot \frac{3}{2} = 0.2241,
\]
CHAPTER # 8

DISCRETE PROBABILITY DISTRIBUTION

\[ P(X=3) = \frac{e^{-3} \cdot 3^3}{3!} = (0.2241)(1) = 0.2241, \text{ and} \]

\[ P(X=4) = \frac{e^{-3} \cdot 4^4}{4!} = (0.2241)\left(\frac{3}{4}\right) = 0.1681, \]

Note that \( e^{-3} = \frac{1}{(2.71828)^3} = 0.0498 \)

(b) Let \( X \) have the Poisson distribution with parameter \( \mu \). Then

\[ P(X=x) = \frac{e^{-\mu} \cdot \mu^x}{x!}, \quad x = 0, 1, 2, \ldots, \infty \]

It is given that

\[ P(X=1) = e^{-\mu} \cdot \mu = 0.3, \text{ and} \]

\[ P(X=2) = \frac{e^{-\mu} \cdot \mu^2}{2!} = 0.2 \]

Dividing the second equation by the first, we get

\[ \frac{\mu}{2!} = \frac{0.2}{0.3}, \text{ or } \mu = 1.33 \]

\[ \therefore P(X=0) = e^{-1.33} = 0.2644, \text{ and} \]

\[ P(X=3) = \frac{e^{-1.33} (1.33)^3}{3!} \]

\[ = \frac{(0.2644)(2.352637)}{6} = 0.1037 \]

To evaluate \( e^{-1.33} \), we let \( y = e^{-1.33} \)

Then \[ \log y = -1.33 \log e = -1.33 (0.4343) \]

\[ = -0.5776 = \bar{1.4223}. \]

\[ \therefore y = \text{Antilog} (\bar{1.4223}) = 0.2644 \]

(c) Let the Poisson distribution be

\[ P(X=x) = \frac{\mu^x \cdot e^{-\mu}}{x!} \text{ for } x = 0, 1, 2, \ldots \]
Given that \( P(X=2) = 3P(X=4) \)

i.e. \( \frac{e^{-\mu} \mu^2}{2!} = 3 \cdot \frac{e^{-\mu} \mu^4}{4!} \), i.e. \( \frac{\mu^2}{2} = \frac{\mu^4}{8} \) giving \( \mu = 2 \).

(i) \( P(X=0) = e^{-2} = 0.1353 = 0.135 \)

(ii) \( P(X\leq3) = \sum_{x=0}^{3} \frac{e^{-2} (2)^x}{x!} \)

\[
= e^{-2} + \frac{e^{-2} \cdot 2}{1!} + \frac{e^{-2} \cdot 4}{2!} + \frac{e^{-2} \cdot 8}{3!}
\]

\[
= e^{-2} \left( 1 + 2 + 2 + \frac{4}{3} \right)
\]

\[
= 0.1353 \cdot 6.3333 = 0.857
\]

8.34. (b) Here \( p = 0.03 \) and \( n = 500 \).

\[
\therefore \text{the mean of the Poisson distribution is }
\]

\[
\mu = np = 500 \times 0.03 = 15.
\]

Let \( X \) denote the number of defective components.

The Poisson distribution then becomes

\[
p(x; 15) = \frac{e^{-15} (15)^x}{x!}, \ x = 0, 1, 2, 3, ...
\]

(i) \( P(X\geq3) = 1 - P(X < 3) \)

\[
= 1 - \left\{ e^{-15} + e^{-15} (15) + \frac{e^{-15} (15)^2}{2!} \right\}
\]

\[
= 1 - e^{-15} \cdot \left( 1 + 15 + 112.5 \right)
\]

\[
= 1 - (0.00000031) (128.5) \ \therefore e^{-15} = 0.00000031
\]

\[
= 1 - 0.000060 = 0.999940
\]

(ii) \( P(X\geq6) = 1 - P(X < 6) \)

\[
= 1 - 0.028298 = 0.971702.
\]

Hence the probability that 2 successive boxes contain 6 or more defective \( = (0.971702)^2 = 0.9442 \).
8.35. (b) Here $p$, the probability of a defective tool = 0.1, so that $q = 0.9$ and $n = 10$.

(i) \[ P(2 \text{ defective tools in } 10) = \binom{10}{2} (0.1)^2 (0.9)^9 = 0.1937 \]

(ii) Let $\mu$ be the mean of the Poisson distribution.

Then \[ \mu = np = 10(0.1) = 1. \]

\[ \therefore P(2 \text{ defective tools in } 10) = \frac{\mu^2 \cdot e^{-\mu}}{2!} = \frac{(1)^2 \cdot e^{-1}}{2!} = \frac{1}{2e} = \frac{1}{2(2.718)} = 0.1839 \]

8.36. (a) Let $X$ denote the number of claims. Then the Poisson distribution with $\mu = 0.05$, is

\[ p(x; 0.05) = \frac{e^{-0.05} (0.05)^x}{x!}, \quad x = 0, 1, 2, 3, \ldots \]

Therefore (i) \[ P(X=0) = e^{-0.05} = 0.9513 \]

(ii) \[ P(1 \text{ or fewer claims}) = P(X \leq 1) = e^{-0.05} + e^{-0.05} (0.05) = 0.9513 + (0.9513)(0.05) = 0.9513 + 0.0476 = 0.9989. \]

(b) Let $p$ be the chance of being killed in an accident.

Then the mean of the Poisson distribution is

\[ \mu = np = 350 \times \frac{1}{1400} = \frac{1}{4} = 0.25 \]

Let $X$ denote the number of fatal accidents. Then

\[ P(\text{At least one fatal accident}) = P(X \geq 1) = 1 - P(X < 1) = 1 - e^{-0.05} (X=0) = 1 - 0.779 = 0.221 \]
8.37. (a) Let $X$ be the r.v. the number of errors per page. Then $X$ has a Poisson distribution with parameter $\mu = 2$. So $P(X=x) = \frac{e^{-2}(2)^x}{x!}$.

Now (i) $P(X\geq 4) = 1 - \sum_{x=0}^{3} \frac{e^{-2}(2)^x}{x!}$, where $e^{-2} = 0.1353$.

$$= 1 - e^{-2} [1 + 2 + 2 + 4/3]$$
$$= 1(0.1353)(6.3333) = 1 - 0.8569 = 0.1431, \text{ and}$$

(ii) $P(X=0) = e^{-2} = 0.1353$.

(b) Given that the number of demands for a car is distributed as a Poisson distribution with mean $\mu = 1.5$.

\[ P(X=x) = \frac{e^{-1.5}(1.5)^x}{x!} \]

where $x$ denotes the number of demands

Proportion of days on which neither car is used

= probability of there being no demand for the car

\[ = \frac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5} = 0.2231 \]

Proportion of days on which some demand is refused

= probability for the number of demands to be more than two.

\[ = 1 - P(X\leq 2), = 1 - \sum_{x=0}^{2} \frac{e^{-1.5}(1.5)^x}{x!} \]

\[ = 1 - \left[ e^{-1.5} + e^{-1.5}(1.5) + \frac{e^{-1.5}(1.5)^2}{2!} \right] \]

\[ = 1 - (0.2231 + 0.3346 + 0.2510) = 1 - 0.8087 = 0.1913. \]

8.38. Let $p$ be the probability of the product being defective. Then $p = 5/100 = 0.05$ and $n = 100$.

The mean of the Poisson distribution, $\mu = np = 0.05 \times 100 = 5$

and the Poisson distribution then becomes
\[ p(x; 5) = \frac{e^{-5} \cdot (5)^x}{x!}, \quad x = 0, 1, 2, \ldots \text{ and } e^{-5} = 0.0067 \]

We seek the probability that a box will fail to meet the guaranteed quality, \textit{i.e.} the probability of there being more than 4 defective pins in a box of 100 pins. In other words, we are to find \( P(X > 4) \).

Thus \( P(X > 4) = 1 - P(X \leq 4) = 1 - \sum_{x=0}^{4} \frac{e^{-5} \cdot (5)^x}{x!} \)

\[ = 1 - e^{-5} \left[ 1 + 5 + \frac{(5)^2}{2} + \frac{(5)^3}{6} + \frac{(5)^4}{24} \right] \]

\[ = 1 - 0.0067 [1 + 5 + 12.5 + 20.83 + 26.04] \]

\[ = 1 - (0.0067 \cdot 65.37) \]

\[ = 1 - 0.4380 = 0.5620. \]

8.39. (b) As the mean and variance of the Poisson distribution are equal, therefore \( \sigma^2 = \mu = 1 \); and the Poisson distribution will be

\[ P(X=x) = \frac{e^{-1} \cdot (1)^x}{x!}. \]

Hence the desired probability is \( P(X=2) = \frac{e^{-1} \cdot (1)^2}{2!} = \frac{1}{2e} = \frac{1}{2(2.7183)} \)

\[ = \frac{1}{5.4366} = 0.1839. \]

8.40. (a) Let \( \mu \) be the parameter of the Poisson distribution.

Then mean = \( \mu \) and \( \sigma = \sqrt{\mu}, \text{ i.e. } \sigma = \sqrt{\text{mean}} \)

In the given statement, mean = 5 and \( \sigma = 4 \)

\[ \therefore \sqrt{\text{mean}}, \text{ i.e. } \sqrt{5} \neq \sigma \]

Hence the given statement is wrong.
(b) Let $\mu$ be the parameter of the Poisson distribution and $N$ be the total number of observations. Then the Poisson distribution is

$$N \cdot e^{-\mu} \cdot \frac{\mu^x}{x!}, \quad x = 0, 1, 2, \ldots, \infty$$

It is given that

$$N \cdot e^{-\mu} = 250, \text{ and } N \cdot e^{-\mu} \cdot \mu = 160.$$ 

Dividing the second equation by the first, we get

$$\mu = \frac{16}{25} = 0.64$$

$$\therefore e^{-0.64} = 1 - (0.64) + \frac{(0.64)^2}{2} - \frac{(0.64)^3}{3!} + \frac{(0.64)^4}{4!} - \ldots$$

$$= 0.5273$$

The frequencies of the next two values are

$$N \cdot e^{-\mu} \cdot \frac{\mu^2}{2!} \text{ and } Ne^{-\mu} \cdot \frac{\mu^3}{3!}.$$ 

Now

$$N \cdot e^{-\mu} \cdot \frac{\mu^2}{2!} = (Ne^{-\mu} \cdot \mu) \left(\frac{\mu}{2}\right) = 160 \times \frac{0.64}{2} = 51, \text{ and}$$

$$Ne^{-\mu} \cdot \frac{\mu^3}{3!} = (Ne^{-\mu} \cdot \frac{\mu^2}{2!}) \left(\frac{\mu}{3}\right) = 51 \times \frac{0.64}{3} = 11.$$ 

8.44. Fitting of the Poisson distribution.

We first calculate the mean of the given distribution and equate it to $\mu$, the parameter of the Poisson distribution $p(x; \mu) = e^{-\mu} \cdot \frac{\mu^x}{x!}$, where $x$ denotes the yeast cell counts.

$$x = \frac{\sum f_i x_i}{\sum f_i} = \frac{0 + 128 + 74 + 54 + 12 + 5}{400} = \frac{273}{400} = 0.6825$$

Therefore $\mu = 0.6825$ and the fitted Poisson distribution is

$$p(x; 0.6852) = \frac{e^{-0.6825} \cdot (0.6825)^x}{x!}, \quad x = 0, 1, 2, 3, \ldots$$
The expected frequencies are obtained by multiplying the probabilities by 400. Thus the expected probabilities and frequencies are calculated as below:

<table>
<thead>
<tr>
<th>$x$</th>
<th>Probability $p(x; 0.6825)$</th>
<th>Expected frequencies $400 \times p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$e^{-0.6825}$</td>
<td>0.5054</td>
</tr>
<tr>
<td>1</td>
<td>$e^{-0.6825}(0.6825)$</td>
<td>0.3449</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{e^{-0.6825}(0.6825)^2}{2!}$</td>
<td>0.1177</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{e^{-0.6825}(0.6825)^3}{3!}$</td>
<td>0.0268</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{e^{-0.6825}(0.6825)^4}{4!}$</td>
<td>0.0046</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{e^{-0.6825}(0.6825)^5}{5!}$</td>
<td>0.0006</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>1.0000</td>
</tr>
</tbody>
</table>

8.45. (a) We first calculate the mean of the given frequency distribution and equate it to (i) $\mu$ and (ii) $np$.

Now $\bar{x} = \frac{\sum fx}{\sum f} = \frac{(0)(531)+(1)(354)+(2)(99)+(3)(15)+(4)(1)}{1000} = \frac{601}{1000} = 0.60$

(i) Therefore $\mu = 0.60$ and the fitted Poisson distribution is

$$P(X=x) = \frac{e^{-0.60}(0.60)^x}{x!} \text{ for } x = 0, 1, 2, 3, ...$$

(ii) Equating to $np$, i.e. $4p$, we get $p = \frac{0.60}{4} = 0.15$. Thus the fitted binomial distribution is

$$P(X=x) = \binom{4}{x} (0.15)^x (0.85)^{4-x} \text{ for } x = 0, 1, 2, 3, 4.$$
The expected frequencies are obtained by multiplying the probabilities by 1000. The probabilities and the expected frequencies are given below:

<table>
<thead>
<tr>
<th>x</th>
<th>(i) Poisson Distribution</th>
<th>(ii) Binomial Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(X=x)$</td>
<td>Expected f</td>
</tr>
<tr>
<td>0</td>
<td>0.5488</td>
<td>549</td>
</tr>
<tr>
<td>1</td>
<td>0.3293</td>
<td>329</td>
</tr>
<tr>
<td>2</td>
<td>0.0988</td>
<td>99</td>
</tr>
<tr>
<td>3</td>
<td>0.0198</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>0.0030</td>
<td>3</td>
</tr>
<tr>
<td>&gt;4</td>
<td>0.0003</td>
<td>0</td>
</tr>
</tbody>
</table>

(b) The mean of the given distribution is

$$
\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{0 + 96 + 68 + 27 + 4 + 0}{440} = \frac{195}{440} = 0.443 = \mu
$$

Thus the fitted Poisson distribution is

$$
p(x; 0.443) = \frac{e^{-0.443} (0.443)^x}{x!}, \quad x = 0, 1, 2, ...
$$

To find the value of $e^{-0.443}$, we let $y = e^{-0.443}$

Taking logs, we get

$$
\log y = -0.443 \log e, \quad \text{where} \quad e = 2.7183
$$

$$
= -0.443 (0.4343) = -0.1924 = -1.8076
$$

$\therefore y = \text{Anti-log} (-1.8076) = 0.6421$

(i) $P(\text{no accident}) = P(X=0) = e^{-0.443} = 0.6421$

(ii) $P(\text{more than one accident}) = P(X>1)$

$$
= 1 - \sum_{x=0}^{\infty} \frac{e^{-0.443} (0.443)^x}{x!}
$$

$$
= 1 - [0.6421 + 0.2845]
$$

$$
= 1 - 0.9266 = 0.0734
$$
8.46. Computation of the frequencies of the fitted Poisson distribution.

Here \( x = \frac{\sum f_i x_i}{\sum f_i} = \frac{0+90+84+36+36+15+6}{300} = \frac{267}{300} = 0.89 \)

Thus the fitted Poisson distribution is

\[ p(x; 0.89) = \frac{e^{-0.89} (0.89)^x}{x!}, \quad x = 0, 1, 2, \ldots \]

To find the value of \( e^{-0.89} \), we put \( y = e^{-0.89} \).

Taking logs, we get

\[ \log y = -0.89 \log e = -0.89 (0.4343) \quad \text{(where } e = 2.718) \]

\[ = -0.3865 \approx 1.6135, \quad \therefore y = \text{Anti-log (1.6135)} = 0.4107 \]

Hence the expected frequencies are computed as below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>Probabilities</th>
<th>Expected ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( e^{-0.89} )</td>
<td>0.4107</td>
</tr>
<tr>
<td>1</td>
<td>( e^{-0.89} \frac{(0.89)^1}{1!} )</td>
<td>0.3655</td>
</tr>
<tr>
<td>2</td>
<td>( e^{-0.89} \frac{(0.89)^2}{2!} )</td>
<td>0.1626</td>
</tr>
<tr>
<td>3</td>
<td>( e^{-0.89} \frac{(0.89)^3}{3!} )</td>
<td>0.0482</td>
</tr>
<tr>
<td>4</td>
<td>( e^{-0.89} \frac{(0.89)^4}{4!} )</td>
<td>0.0107</td>
</tr>
<tr>
<td>5</td>
<td>( e^{-0.89} \frac{(0.89)^5}{5!} )</td>
<td>0.0019</td>
</tr>
<tr>
<td>6</td>
<td>( e^{-0.89} \frac{(0.89)^6}{6!} )</td>
<td>0.0003</td>
</tr>
</tbody>
</table>

8.47. The mean number of accidents a day is

\[ x = \frac{\sum f_i x_i}{\sum f_i} = \frac{0+113+128+63+28+15+6+7}{300} = \frac{360}{300} = 1.2 \]
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Thus the fitted Poisson distribution is

\[ p(x; 1.2) = \frac{e^{-1.2} (1.2)^x}{x!} \]

To evaluate \( e^{-1.2} \), we let \( y = e^{-1.2} \)

Then \( \log y = -1.2 \log e \)

\[ = -1.2 (0.4343) = -0.52116 = \bar{1}.47884 \]

\[ \therefore y = \text{Anti-log} (\bar{1}.47884) = 0.3011. \]

The expected frequencies are calculated as below:

<table>
<thead>
<tr>
<th>Accidents per day (x)</th>
<th>Probabilities ( p(x; 1.2) )</th>
<th>Expected ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( e^{-1.2} ) = 0.3011</td>
<td>90.3</td>
</tr>
<tr>
<td>1</td>
<td>( e^{-1.2} \cdot \frac{(1.2)}{1!} = (0.3011)(1.2) = 0.3613 )</td>
<td>108.4</td>
</tr>
<tr>
<td>2</td>
<td>( e^{-1.2} \cdot \frac{(1.2)^2}{1!} = (0.3613)(0.6) = 0.2168 )</td>
<td>65.0</td>
</tr>
<tr>
<td>3</td>
<td>( e^{-1.2} \cdot \frac{(1.2)^3}{3!} = (0.2168)(0.4) = 0.0867 )</td>
<td>26.0</td>
</tr>
<tr>
<td>4</td>
<td>( e^{-1.2} \cdot \frac{(1.2)^4}{4!} = (0.0867)(0.3) = 0.0260 )</td>
<td>7.8</td>
</tr>
<tr>
<td>5</td>
<td>( e^{-1.2} \cdot \frac{(1.2)^5}{5!} = (0.0260)(0.24) = 0.0062 )</td>
<td>1.9</td>
</tr>
<tr>
<td>6</td>
<td>( e^{-1.2} \cdot \frac{(1.2)^6}{6!} = (0.0062)(0.2) = 0.0012 )</td>
<td>0.4</td>
</tr>
<tr>
<td>7+</td>
<td>( 1 - \sum_{x=0}^{6} \frac{e^{-1.2}(1.2)^x}{x!} ) = 0.0007</td>
<td>0.2</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>( = 1.0000 ) = 300.0</td>
<td></td>
</tr>
</tbody>
</table>

And \( P(4 \text{ or more accidents}) = 1 - P(\text{less than 4 accidents}) \)

\[ = 1 - (0.3011 + 0.3613 + 0.2168 + 0.0867) \]

\[ = 1 - 0.9659 = 0.0341. \]
8.48. (b) Let $X$ denote the number of customers entering the shop in $t$ units of time.

Then we have

\[ \lambda = 30 \text{ persons per hour} = 30 \text{ persons per 60 minutes} \]

As 2 minutes is \( \frac{2}{60} = \frac{1}{30} \) units of time, so \( t = \frac{1}{30} \) and therefore the average number of persons per 2 minutes interval, i.e. \( \lambda t = 30 \times \frac{1}{30} = 1 \).

Hence, using the Poisson process formula

\[
p(x; \lambda t) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}, \quad \text{we get}
\]

\[
p(0; 1) = \frac{e^{-1} \cdot (1)^0}{0!} = e^{-1} = \frac{1}{2.7183} = 0.3679.
\]

8.49. (a) Taking 50 square feet as the unit of area, we have

\[ \lambda = 1 \text{ flaw per 50 square feet}. \]

As 32 square feet are \( \frac{32}{50} = 0.64 \) units of area, so \( t = 0.64 \), and therefore the average number of flaws per 32 square feet, i.e. \( \lambda t = 1 \times 0.64 = 0.64 \). Assuming the flaws are a Poisson process, we have

\[
P(\text{no flaw occurs in 32 square feet}) = \frac{e^{-0.64}(0.64)^0}{0!} = 0.5272
\]

\[
P(\text{atmost one flaw occurs in 32 square feet}) = \sum_{x=0}^{1} \frac{e^{-0.64}(0.64)^x}{x!}
\]

\[
= e^{-0.64} + e^{-0.64} \cdot 0.64
\]

\[
= 0.5272 + 0.3374
\]

\[
= 0.8646
\]

(b) Taking 12 hours from 9 p.m. until 9 a.m. the next morning as the unit of time, we have \( \lambda = 3 \) calls per 12 hours
As 6 hours from midnight to 6 a.m. are \( \frac{6}{12} = \frac{1}{2} \) units of time, so \( t = \frac{1}{2} \) and therefore the average number of calls per 6 hours, i.e.
\[
\lambda t = 3 \times \frac{1}{2} = 1.5.
\]
As the arrivals of calls are assumed to be Poisson process, therefore we have
\[
P(\text{doctor receives no call in 6 hours}) = \frac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5} = 0.2231
\]

8.50. Taking 25 days as unit of time, we have
\[
\lambda = 1 \text{ breakdown in 25 days.}
\]
As 10 days is \( \frac{10}{25} = 0.4 \) unit of time, so \( t = 0.4 \) and therefore the average number of breakdowns per 10 days, i.e. \( \lambda t = 1 \times 0.4 = 0.4 \).

Hence, using the Poisson process formula \( p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \), we have
(i) \( P(\text{exactly one breakdown in 10 days}) = e^{-0.4} \frac{(0.4)^1}{1!} = (0.6704)(0.4) = 0.2682 \)
(ii) \( P(\text{more than one breakdown in 10 days}) = 1 - P(0 \text{ or } 1 \text{ breakdown}) = 1 - \sum_{x=0}^{1} \frac{e^{-0.4}(0.4)^x}{x!} = 1 - [0.6704 + 0.2682] = 1 - 0.9386 = 0.0614. \)

8.51. Taking 60 minutes, (10 to 11 A.M.) as unit of time, we have
\[
\lambda = 300 \text{ cars passing in 60 minutes.}
\]
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As 1 minute is \( \frac{1}{60} \) unit of time, so \( t = \frac{1}{60} \) and therefore the average number of cars passing per minute, \( i.e. \lambda t = 300 \times \frac{1}{60} = 5 \)

Hence, using the Poisson process formula \( p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \),

we have (i) \( P(\text{not more than 4 cars in 1-minute}) = \sum_{x=0}^{4} \frac{e^{-5} \cdot (5)^x}{x!} \)

\[ = e^{-5} \left[ 1 + 5 + \frac{(5)^2}{2} + \frac{(5)^3}{6} + \frac{(5)^4}{24} \right] \]

\[ = (0.0067) \left[ 1 + 5 + 12.5 + 20.83 + 26.04 \right] \]

\[ = (0.0067) \cdot (65.37) \quad (\because e^{-5} = 0.0067) \]

\[ = 0.4380 \]

(ii) \( P(5 \text{ or more cars pass in 1-minute}) \)

\[ = 1 - P(\text{less than 5 cars pass}) \]

\[ = 1 - \sum_{x=0}^{4} \frac{e^{-5} \cdot (5)^x}{x!} = 1 - 0.4380 = 0.5620. \]

8.53. (b) Let \( N \) be equal to number of eggs laid by an insect. Then \( N \) is assumed to have a Poisson distribution with mean \( m \).

It is given that the probability of an egg developing is \( p \).

If \( X \) denotes the number of eggs that develop, then for given \( N=n \), the variate \( X \) has a binomial distribution based on \( n \) repetitions with parameter \( p \). That is

\[ P(X=k \mid N=n) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, ..., n \]

Hence according to the theorem of total probability, we have

\[ P(X=k) = \sum_{n=k}^{\infty} P(X=k \mid N=n) \cdot P(N=n) \]

\[ = \sum_{n=k}^{\infty} \binom{n}{k} p^k q^{n-k} \cdot \frac{e^{-m} \cdot m^n}{n!} \]
\[
\frac{e^{-m} p^k}{k!} \sum_{n=k}^{\infty} \frac{m^n q^{n-k}}{(n-k)!} = \frac{e^{-m} m^k}{k!} \sum_{n=k}^{\infty} \frac{(mq)^{n-k}}{(n-k)!} \\
= \frac{e^{-m} (mp)^k}{k!} \sum_{j=0}^{\infty} \frac{(mj)^{n-k}}{j!} \quad \text{where} \quad j = n-k \\
= \frac{e^{-m} (mp)^k}{k!} e^{mq} = \frac{e^{-m} (mp)^k e^{m(1-p)}}{k!} = \frac{e^{-mp} (mp)^k}{k!},
\]

which is a Poisson distribution with mean \( mp \).

8.54. (b) We observe that

(i) there are two possible outcomes, i.e. the swimmer will or will not cross the lake,

(ii) probability of success in swimming across the lake for each swimmer is \( p = 0.4 \),

(iii) the swimmers will cross the lake independently, and

(iv) the experiment is repeated 10 times to get fourth success.

Therefore the negative binomial distribution with \( k = 4 \) (4th success) and \( x = 10 \), is appropriate. Hence

\[
b^\ast(10; 4, 0.4) = \binom{10-1}{4-1} (0.4)^4 (0.6)^{10-4} = 84 \cdot (0.4)^4 (0.6)^6 = 0.10033
\]

8.55. (b) Here \( E(X) = 10 \), and \( \sigma^2 = 9 \).

Since mean is greater than variance, so \( X \) cannot have a negative binomial distribution.

(c) Here \( p \), the probability of getting a head = \( \frac{1}{2} \),

\( k = 3 \) (third success), and \( x = 7 \) (seventh trial).

\[
b^\ast(7; 3, \frac{1}{2}) = \binom{7-1}{3-1} \left( \frac{1}{2} \right)^3 \left( \frac{1}{2} \right)^{7-3} = 15 \cdot \left( \frac{1}{2} \right)^7 = \frac{15}{128} = 0.1172.
\]
8.56. (b) Here \( p \), the probability of getting installed a black telephone = 0.3.

\[
k = 5 \text{ (5th success), and } x = 10 \text{ (10th telephone)}
\]

\[
\therefore \quad b^*(10; 5, 0.3) = \binom{10-1}{5-1} (0.3)^5 (0.7)^{10-5}
\]

\[
= 126 (0.3)^5 (0.7)^5 = 0.0515
\]

8.57. (b) The Negative Binomial distribution gives the probability that the \( k \)th success occurs on the \( x \)th trial for \( x = k, k+1, k+2, \ldots \). Its probability function is

\[
p(x) = \binom{x-1}{k-1} p^k q^{x-k}, \quad \text{where } x = k, k+1, k+2, \ldots
\]

It has two parameters \( k \) and \( p > 0 \). To show that the probabilities add to 1, we proceed as below:

\[
\sum p(x) = \sum_{x=k}^{\infty} \binom{x-1}{k-1} p^k q^{x-k}, \quad (x = k, k+1, k+2, \ldots)
\]

Let \( y = x - k \). Then

Sum of prob = \[
\sum_{y=0}^{\infty} \binom{y+k-1}{k-1} p^k q^y, \quad (y = 0, 1, 2, \ldots)
\]

\[
= p^k \sum_{y=0}^{\infty} \binom{y+k-1}{k-1} q^y
\]

\[
= p^k \left[ \binom{k-1}{k-1} q^0 + \binom{k}{k-1} q + \binom{k+1}{k-1} q^2 + \ldots \right]
\]

\[
= p^k \left[ 1 + kq + \frac{k(k+1)}{2!} q^2 + \ldots \right]
\]

\[
= p^k (1 - q)^{-k} = p^k p^{-k} = 1.
\]

The m.g.f. about the origin is

\[
M_0(t) = E(e^{ty}) = p^k \sum_{y=0}^{\infty} \binom{y+k-1}{k-1} q^y \cdot e^{ty}
\]
\[
\sum_{y=0}^{\infty} \binom{y+k-1}{k-1} (q e^t)^y = p^k (1 - q e^t)^{-k}
\]

\[
= \left[ \frac{p}{1 - q e^t} \right]^k = \left[ \frac{p}{p + q - q e^t} \right]^k
\]

\[
= \left[ \frac{1}{1 - q (e^t - 1)} \right]^k
\]

\[
= [1 - \lambda (e^t - 1)]^{-k}, \quad \text{where} \quad \lambda = q/p.
\]

Taking logs, we have

\[
\kappa(t) = \log M_0(t) = -k \log [1 - \lambda (e^t - 1)]
\]

\[
= k \left[ \lambda (e^t - 1) + \frac{1}{2} \lambda^2 (e^t - 1)^2 + \frac{1}{3} \lambda^3 (e^t - 1)^3 + \frac{1}{4} \lambda^4 (e^t - 1)^4 + \ldots \right]
\]

\[
= k \left[ \lambda (t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots) + \frac{1}{2} \lambda^2 t^2 (1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots)^2
\]

\[
+ \frac{1}{3} \lambda^3 t^3 (1 + \frac{t}{2!} + \ldots)^3 + \frac{1}{4} \lambda^4 t^4 + \ldots \right]
\]

\[
= k \left[ \lambda t + (\lambda + \lambda^2) \frac{t^2}{2!} + (\lambda + 3 \lambda^2 + 2 \lambda^3) \frac{t^3}{3!}
\]

\[
+ (\lambda + 7 \lambda^2 + 12 \lambda^3 + 6 \lambda^4) \frac{t^4}{4!} + \ldots \right]
\]

\[
= k \left[ \frac{q}{p} t + \frac{q}{p^2} \frac{t^2}{2!} + \frac{q(1+q)}{p^3} \frac{t^3}{3!} + \frac{q(1+4q+q^2)}{p^4} \frac{t^4}{4!} + \ldots \right]
\]

denoting co-efficients, we get

\[
\kappa_1 = \frac{k q}{p}, \quad \kappa_2 = \frac{k q}{p^2}, \quad \kappa_3 = \frac{k q (1+q)}{p^3}, \quad \text{and}
\]

\[
\kappa_4 = \frac{k q (1+4q+q^2)}{p^4}
\]

8.58. (c) Let \(X\) denote the number of trials required to get the occurrence of 1st head.

The head occurs on the third trial, \(i.e.\) the first success occurs on the 3rd trial, therefore the geometric distribution with \(p = 1/2\) and \(x = 3\) is used.
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Hence, using the geometric distribution \( P(X=x) = pq^{x-1} \),
we get
\[
P(X=3) = \left(\frac{1}{2}\right)^2 \frac{1}{2} = \frac{1}{8}.
\]

8.59. (b) Sample space \( S \) associated with this experiment contains \( \binom{52}{13} \) sample points.

Let \( A \) be the event that a bridge hand contains 5 spades, 2 hearts, 3 diamonds and 3 clubs.

Then \( A \) contains \( \binom{13}{5} \cdot \binom{13}{2} \cdot \binom{13}{3} \cdot \binom{13}{3} \) sample points.

Hence \( P(A) = \frac{\binom{13}{5} \binom{13}{2} \binom{13}{3} \binom{13}{3}}{\binom{52}{13}} = \frac{(1287)(78)(286)(286)}{635,013,559,600} = 0.0129. \)

This question actually relates to the extension of the hypergeometric distribution where the population is partitioned into 4 cells.

8.60. (a) Here \( P_1 = P(\text{bulb is red}) = 0.50, \)
\( P_2 = P(\text{bulb is blue}) = 0.30, \)
\( P_3 = P(\text{bulb is green}) = 0.20, \) and
\( x_1 = 2, x_2 = 1 \) and \( x_3 = 2. \)

\[ \therefore P(2 \text{ are red, } 1 \text{ is green, } 2 \text{ are blue}). \]
\[
= \frac{5!}{2! \cdot 1! \cdot 2!} (0.5)^2 (0.3) (0.2)^2 \\
= \frac{30 \times 3 \times 4}{4 \times 10 \times 100} = \frac{9}{100} = 0.09. \]

(b) Here \( p_1 = P(\text{red marble is drawn}) = \frac{5}{10}, \)
\( p_2 = P(\text{white marble is drawn}) = \frac{3}{10}. \)
\[ p_3 = P(\text{blue marble is drawn}) = \frac{2}{10}, \text{ and} \]

(i) \( x_1 = 3, x_2 = 2 \) and \( x_3 = 1. \)

Thus \( P(3 \text{ are red, 2 are white, 1 is blue}). \)

\[
\begin{align*}
&= \frac{6!}{3! \cdot 2! \cdot 1!} \left( \frac{5}{10} \right)^3 \left( \frac{3}{10} \right)^2 \left( \frac{2}{10} \right)^1 \\
&= \frac{60 \times 1 \times 9 \times 2}{8 \times 100 \times 10} \quad = \frac{27}{200} = 0.135.
\end{align*}
\]

(ii) \( x_1 = 2, x_2 = 3 \) and \( x_3 = 1. \)

Thus \( P(2 \text{ are red, 3 are white, 1 is blue}). \)

\[
\begin{align*}
&= \frac{6!}{2! \cdot 3! \cdot 1!} \left( \frac{5}{10} \right)^2 \left( \frac{3}{10} \right)^3 \left( \frac{2}{10} \right)^1 \\
&= \frac{60 \times 27 \times 2}{4 \times 1000 \times 10} \quad = \frac{81}{1000} = 0.081.
\end{align*}
\]

(iii) \( x_1 = 2, x_2 = 2 \) and \( x_3 = 2. \)

Thus \( P(2 \text{ of each colour appear}) \)

\[
\begin{align*}
&= \frac{6!}{2! \cdot 2! \cdot 2!} \left( \frac{5}{10} \right)^2 \left( \frac{3}{10} \right)^2 \left( \frac{2}{10} \right)^2 \\
&= \frac{90 \times 9 \times 4}{4 \times 100 \times 100} \quad = \frac{81}{1000} = 0.081.
\end{align*}
\]